

Note on the equations of diffusion operators associated to a positive matrix

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ABSTRACT

In this paper, we describe the general framework to describe the diffusion operators associated to a positive matrix. We define the equations associated to diffusion operators and present some general properties of their state vectors. We show how this can be applied to prove and improve the convergence of a fixed point problem associated to the matrix iteration scheme. The approach can be understood as a decomposition of the matrix-vector product operation in elementary operation at the vector entry level.

Categories and Subject Descriptors

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General Terms

Algorithms, Performance

Keywords

Numerical computation; Iteration; Fixed point; Eigenvector.

1. INTRODUCTION

In this paper, we assume that the readers are already familiar with the idea of the fluid diffusion associated to the D-iteration [2] to solve the equation:

$$X = P.X + B$$

and its application to PageRank equation [3].

For the general description of alternative or existing iteration methods, one may refer to [1, 5].

2. NOTATION

We will use the following notations:

- $P \in (\mathbb{R}^+)^{N \times N}$ a positive matrix;
- $I_d \in (\mathbb{R}^+)^{N \times N}$ the identity matrix;
- J_i the matrix with all entries equal to zero except for the i -th diagonal term: $(J_i)_{ii} = 1$;
- $I = \{i_1, i_2, \dots, i_n, \dots\}$ the sequence of nodes for the diffusion: $i_k \in \{1, \dots, N\}$;

- $\sigma_v : \mathbb{R}^N \rightarrow \mathbb{R}$ the scalar product for a given strictly positive vector $V > 0$: $\sigma_v(X) = \langle V | X \rangle = \sum_{i=1}^N v_i x_i$;
- e the normalized unit column vector $1/N(1, \dots, 1)^t$.

We say that P is σ_v -decreasing if:

$$\forall X \in (\mathbb{R}^+)^N, \sigma_v(PX) \leq \sigma_v(X).$$

We define $P^\alpha = (1 - \alpha)I_d + \alpha P$.

Then, we have the following results:

THEOREM 1. σ_v -decreasing property is stable by composition of operators (matrix product).

If P is σ_v -decreasing, for $\alpha \geq 0$, P^α is σ_v -decreasing.

If P is σ_v -decreasing, for $(\alpha, \alpha') \in (\mathbb{R}^+)^2$ such that $\alpha \leq \alpha'$, $\sigma_v(P^{\alpha'} X) \leq \sigma_v(P^\alpha X)$.

PROOF. The first point is obvious. The other points are based on the linearity of σ_v .

2.1 Diffusion operators

We define the N diffusion operators associated to P by:

$$P_i = I_d - J_i + P.J_i$$

THEOREM 2. If P is σ_v -decreasing, then the diffusion operators P_i are σ_v -decreasing. Therefore, for $\alpha \geq 0$,

$$P_i^\alpha = I_d + \alpha(P - I_d).J_i$$

is σ_v -decreasing.

PROOF. $\sigma_v(P_i.X) = \sigma_v(X) + \sigma_v(P.J_i.X) - \sigma_v(J_i.X)$ and we have $\sigma_v(P.J_i.X) \leq \sigma_v(J_i.X)$, therefore $\sigma_v(P_i.X) \leq \sigma_v(X)$. The last point is the application of Theorem 1 to P_i .

We recall that the D-iteration is defined by the couple $(P, B) \in \mathbb{R}^{N \times N} \times \mathbb{R}^N$ and exploits two state vectors: H_n (history) and F_n (residual fluid):

$$F_0 = B \quad (1)$$

$$F_n = P_{i_n} F_{n-1} \quad (2)$$

and

$$H_0 = 0 = (0, \dots, 0)^t \quad (3)$$

$$H_n = H_{n-1} + J_{i_n} F_{n-1}. \quad (4)$$

The D-iteration is the joint iteration on (F_n, H_n) .

Now, we consider a bit more general diffusion iterations as follows:

$$F_n^\alpha = P_{i_n}^{\alpha_n} F_{n-1}^\alpha. \quad (5)$$

and

$$H_n^\alpha = H_{n-1}^\alpha + \alpha_n J_{i_n} F_{n-1}^\alpha \quad (6)$$

where $\alpha_n \geq 0$. If for all n , $\alpha_n = 1$, we have the usual diffusion iteration.

THEOREM 3. (F_n^α, H_n^α) satisfies:

$$H_n^\alpha + F_n^\alpha = P.H_n^\alpha + B. \quad (7)$$

PROOF. The proof is the same as for the $\alpha = 1$ (cf. [3]) diffusion equations by induction and using equations (5) and (6).

THEOREM 4. Assume we choose $F_0 \geq 0$ and $H_0 = 0$. Then, F_n^α and H_n^α are positive and $(H_n^\alpha)_i$ is an increasing function for all i .

If P is σ_v -decreasing, then $\sigma_v(F_n^\alpha)$ is a decreasing function.

PROOF. The proof is straightforward.

THEOREM 5. If we build the two diffusion iterations (F_n^α, H_n^α) and $(F_n^{\alpha'}, H_n^{\alpha'})$ from the same initial vector F_0 ($H_0 = 0$) and for the same diffusion sequence I , if for all n , $0 \leq \alpha_n \leq \alpha'_n \leq 1$, then we have:

- $\sigma_v(F_n^{\alpha'}) \leq \sigma_v(F_n^\alpha)$;
- $H_n^{\alpha'} \geq H_n^\alpha$ (for each vector entry);
- $H_n^{\alpha'} + F_n^{\alpha'} \geq H_n^\alpha + F_n^\alpha$.

PROOF. The first inequality is a direct consequence of Theorem 2 and Theorem 1. The third is a direct consequence of the second inequality using Theorem 3. For the second inequality, we prove by induction: we have obviously $H_1^{\alpha'} \geq H_1^\alpha$. assume we have, $H_n^{\alpha'} \geq H_n^\alpha$. Then, from (6):

$$\begin{aligned} H_{n+1}^{\alpha'} &= H_n^{\alpha'} + \alpha'_{n+1} J_{i_{n+1}} F_n^{\alpha'} \\ &\geq H_n^{\alpha'} + \alpha_{n+1} J_{i_{n+1}} F_n^{\alpha'} \end{aligned}$$

and

$$H_{n+1}^\alpha = H_n^\alpha + \alpha_{n+1} J_{i_{n+1}} F_n^\alpha$$

We need to prove that:

$$H_n^{\alpha'} - H_n^\alpha \geq \alpha_{n+1} J_{i_{n+1}} (F_n^\alpha - F_n^{\alpha'}).$$

For $i \neq i_{n+1}$, $(H_n^{\alpha'} - H_n^\alpha)_i \geq 0$ and $(J_{i_{n+1}}(F_n^\alpha - F_n^{\alpha'}))_i = 0$. For $i = i_{n+1}$, we only need to handle the case $(F_n^\alpha - F_n^{\alpha'})_{i_{n+1}} \geq 0$. We use the relation: $H_n^{\alpha'} - H_n^\alpha \geq F_n^\alpha - F_n^{\alpha'}$ to get the inequality.

REMARK 1. The power iteration $X_n = P.X_{n-1}$ can be described in the above scheme (F_n^α, H_n^α) with $X_n = F_{Nn}^\alpha$ taking $F_0 = X_0$ and if we apply the cyclic sequence $1, \dots, N$ ($i_n = n \bmod N$) where $\alpha_{kN+i} \leq 1$ is chosen such that we diffuse exactly $(P^k.X_0)_i$ (such a value exists and is less than 1 because after the diffusion of nodes $1, \dots, i$ the residual fluid on $(i+1)$ -th node can only be increased).

2.2 Application to DI+

The D-iteration adaptation to the general case (DI+) has been presented in [4]. For the sake of simplicity, we show how the above results apply to the DI+ only for the case when P is an irreducible ergodic positive matrix having 1 as spectral radius. First, we take for v the left-eigenvector of P such that $v^t.P = v$.

DI+'s idea is to apply the diffusion process to $(P, P.e - e)$.

THEOREM 6. If we choose the sequence of nodes I such that we only diffuse negative fluids, then the diffusion applied on $(P, P.e - e)$ converges to $X - e$.

PROOF. The diffusion from $P.e - e$ can be decomposed as the difference of two diffusion process $(F_n^{\alpha'}, H_n^{\alpha'})$ and (F_n^α, H_n^α) as follows: we start with $F_0 = e$. For the N first diffusions, we choose $i_n = n$ and

- for $P_{i_n}^{\alpha_n}$, $\alpha_n = 0$;
- for $P_{i_n}^{\alpha'_n}$, α'_n such that we diffuse exactly $1/N$ (such a value exists and is less than 1 because after the diffusion of nodes $1, \dots, i$ the residual fluid on $(i+1)$ -th node can only be increased).

Then we have: $F_N^\alpha = e$, $H_N^\alpha = 0$ and $F_N^{\alpha'} = P.e$, $H_N^{\alpha'} = e$. Then from the $(N+1)$ -th diffusion, we apply exactly the same sequence with $\alpha_n = \alpha'_n = 1$. From Theorem 5, we have $H_n + e = H_n^{\alpha'} - H_n^\alpha \geq 0$ and we have $\sigma_v(F_n) = \sigma_v(F_n^{\alpha'} - F_n^\alpha) = 0$. If we only diffuse negative fluids, this means that H_n is a decreasing function (per entry). Since we have $0 \leq H_n + e \leq e$, H_n is convergent.

It has been shown in [4] that $|F_n|_v = \sum_i |v_i \times (F_n)_i|$ is a decreasing function. The convergence of H_n implies of course the convergence to zero of F_n .

REMARK 2. We observed that the implicit strategy on DI+ (diffusion of negative fluids) can easily bring a computation time reduction factor above 10.

REMARK 3. If we mix the diffusion of positive and negative fluids, there is no guarantee that the algorithm DI+ converges and there is snake-configuration example to prove that we may oscillate ($1 \rightarrow 2$ and $1 \rightarrow 3$; $2 \rightarrow 4$; $3 \rightarrow 5$; $4 \rightarrow 1$; $5 \rightarrow 1$). Now one can easily understand that the optimal sequence may require the diffusion of positive fluids (in the above snake-configuration, the optimal choice is choosing once the node 1 which has a positive fluid assuming a weight of 0.5 on $1 \rightarrow 2$ and $1 \rightarrow 3$). If P is irreducible and ergodic, the author conjectures that we still have the convergence with $H_n + e \leq 1$, if one choose only positive fluids. The DI+ algorithm should converge when P is not ergodic. The DI+ algorithm should even converge when P is not irreducible under the condition that we only take negative fluids.

3. CONCLUSION

We presented the general equations of the diffusion operators and the general properties of the associated state vectors with an illustration of the application to prove the convergence of DI+ with a new sequence I choice strategy.

4. REFERENCES

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